LEBANESE AMERICAN UNIVERSITY
DEPARTMENT OF COMPUTER SCIENCE AND MATHEMATICS
MTH 304 – DIFFERENTIAL EQUATIONS
EXAM 3 (OPTIONAL) – SPRING 2014

NAME: KEY. **ID #:**

THIS EXAM CONSISTS OF 6 PAGES AND 2 PROBLEMS. THE LAST PAGE MAY BE USED FOR SCRAP WORK. ANSWER THE QUESTIONS IN THE SPACES PROVIDED; IF MORE SPACE IS NEEDED, USE THE BACK OF THE PAGES. THE TIME OF THE EXAM IS 60 MINUTES.

Question Number	Grade
1. (32%)	
2. (18%)	
TOTAL	

1. Use **Laplace Transforms** to solve completely the following second-order non-homogeneous differential equations:
   a. **(16%)** $y''-2y'+2y=0; y(0)=0, y'(0)=1.$

$$\mathcal{L}(y'') - 2\mathcal{L}(y') + 2\mathcal{L}(y) = \mathcal{L}(0)$$

$$\Rightarrow \left[s^2 Y - s\underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=1}\right] - 2\left[sY - \underbrace{y(0)}_{=0}\right] + 2Y = 0; \; Y = \mathcal{L}(y)$$

$$\Rightarrow s^2 Y - 1 - 2sY + 2Y = 0$$

$$\Rightarrow \cancel{Y(s^2 - 2s) + 1) = 0}$$

$$\Rightarrow Y(s^2 - 2s + 2) = 1$$

$$\Rightarrow Y = \frac{1}{s^2 - 2s + 2} = \frac{1}{(s^2 - 2s + 1) + 1}$$

$$\Rightarrow Y = \frac{1}{(s-1)^2 + 1}$$

$$\therefore y = \mathcal{L}^{-1}\left[\frac{1}{(s-1)^2+1}\right]$$

$$\Rightarrow \boxed{y = e^t \sin t}$$

b. (16%) $y''-6y'+9y=t^2e^{3t}; y(0)=2; y'(0)=6.$

$$\mathcal{L}(y'') - 6\mathcal{L}(y') + 9\mathcal{L}(y) = \mathcal{L}(t^2e^{3t})$$

$$[s^2Y - s\underbrace{y(0)}_{=2} - \underbrace{y'(0)}_{=6}] - 6[sY - \underbrace{y(0)}_{2}] + 9Y = (-1)^2\frac{d^2}{ds^2}\mathcal{L}(e^{3t})$$

$$\Rightarrow s^2Y - 2s - 6 - 6sY + 12 + 9Y = \cancel{1}\ \cancel{9e}\ \frac{d^2}{ds^2}\left(\frac{1}{s-3}\right)$$

$$\Rightarrow Y(s^2-6s+9) - 2(s-3) = \frac{2}{(s-3)^3}$$

$$\Rightarrow Y(s-3)^2 - 2(s-3) = \frac{2}{(s-3)^3}$$

$$\Rightarrow Y(s-3)^2 = \frac{2}{(s-3)^3} + 2(s-3) \cancel{= \frac{2+2(s-3)^4}{(s-3)^3}}$$

$$\Rightarrow Y = \frac{2}{(s-3)^5} + \frac{2}{(s-3)}.$$

$$\Rightarrow y = 2\mathcal{L}^{-1}\left[\frac{1}{(s-3)^5}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{s-3}\right]$$

$$= \frac{2}{24}\mathcal{L}^{-1}\left[\frac{24}{(s-3)^5}\right] + 2e^{3t}$$

$$= \frac{1}{12}t^4e^{3t} + 2e^{3t}$$

$$\left[\frac{d^4}{ds^4}\left(\frac{1}{s-3}\right) = \frac{24}{(s-3)^5}\right].$$

2. Consider the ordinary differntial equation $y''-xy=0$. We wish to express its solution in series form: $\sum_{n=0}^{\infty} c_n x^n$

a) **(5%)** Show $c_2 = 0$

b) **(10%)** Express c_3, c_4, c_5, c_6, c_8 in terms of c_0, c_1, c_2.

$$y = \sum_{n=0}^{\infty} c_n x^n \Rightarrow y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0.$$

$$\Rightarrow \cancel{\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - \sum_{n=0}^{\infty} c_n x^{n+1}}$$

$$\sum_{n=1}^{\infty} c_n x^n \rightarrow \sum_{n=2}^{\infty} n c_n x^{n-1} \rightarrow \sum_{n=3}^{\infty} n(n-1) c_n x^{n-2}$$

$$xy \rightarrow \sum_{n=1}^{\infty} c_n x^{n+1} \rightarrow \sum x^n$$

$$\sum_{n=0} (n+2)(n+1) c_{n+2} x^n - \sum_{n=1}^{\infty} c_{n-1} x^n$$

$$\Rightarrow 2c_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) c_{n+2} - c_{n-1} \right] x^n = 0$$

$$\Rightarrow c_2 = 0.$$

$$c_{n+2} = \frac{c_{n-1}}{(n+2)(n+1)};$$

$n=1 \rightarrow c_3 = \boxed{\frac{c_0}{3 \cdot 2}}$

$n=2 \rightarrow c_4 = \frac{c_1}{4 \cdot 3}$

$n=3 \rightarrow c_5 = \frac{c_2}{5 \cdot 4} = 0$

$n=4 \Rightarrow c_6 = \frac{c_3}{6 \cdot 5} = \boxed{\frac{c_0}{6 \cdot 5 \cdot 3 \cdot 2}}$

$n=5 \rightarrow c_7 = \frac{c_4}{7 \cdot 6} = \frac{c_1}{7 \cdot 6 \cdot 4 \cdot 3}$

$n=6 \rightarrow c_8 = \frac{c_5}{8 \cdot 7} = 0.$

c) **(3%)** Complete the following:

i. $c_{3k} = \cancel{(-1)^k} \dfrac{?}{(3k)!}$

ii. $c_{3k+1} = \cancel{(-1)^k} \dfrac{?}{(3k+1)!}$

iii. $c_{3k+2} = ?$

$c_{3k+2} = 0$

$$c_{3k} = \frac{c_0 \cdot (3k-2)(3k-5) \cdots 4}{(3k)!}$$

$$c_{3k+1} = \frac{c_1 \cdot (3k-1)(3k-4) \cdots 2}{(3k+1)!}$$

Scratch